On N-high subgroups of Abelian p-groups

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Let G be an Abelian p-group, and N be a subgroup of G. Then a subgroup H of G maximal with respect to disjointness from N will be called an N-high subgroup of G. Let G^1 be the subgroup of all elements of infinite height in G, namely, $G^1=\bigcap_{n<\omega} p^nG$ where ω is a first infinite ordinal. If $N=G^1$, a G^1-high subgroup H will be called a high subgroup of G. J. M. Irwin showed in [2] that high subgroups of Abelian torsion groups are pure. J. M. Irwin and E. A. Walker generalized this result in [3] as follows: If G is a torsion group, and if N is any subgroup of G^1, then any N-high subgroup of H is pure in G. Another proof was given in P. Hill and C. Megibben [1].

For any ordinal α, $p^\alpha G$ is defined inductively by letting $p^0G=G$, $p^\alpha G=p(p^\beta G)$ if $\beta+1=\alpha$, and $p^\alpha G=\bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal. In this note, we prove the next theorem.

THEOREM. *Assume that G is an Abelian p-group and that α is any ordinal. If N is any subgroup of $p^\alpha G$, and if H is an N-high subgroup of G, then we have*

$$p^\beta H=H\cap p^\beta G$$

for every ordinal $\beta\leqq\alpha+1$.

PROOF. We use a transfinite induction on β.

If $\beta=0$, it holds obviously $p^0H=H\cap p^0G$.

Take an ordinal $\gamma(0<\gamma\leqq\alpha+1)$, and assume the statement true for every ordinal $\beta<\gamma$. We prove it for the ordinal γ.

If $\gamma=\beta+1$, it suffices to show that $a\in H\cap p^\gamma G=H\cap p(p^\beta G)$ implies $a\in p(p^\beta H)=p^\gamma H$. An element a of $H\cap p^\gamma G$ has the form $a=px$ where x is an element of $p^\beta G$. We may assume that x does not belong to H. For, if x is an element of H, then x belongs to $H\cap p^\beta G=p^\beta H$. Therefore, a is an element of $p(p^\beta H)=p^\gamma H$, and there is nothing to prove in this case. Now, $\{H, x\}$ intersects with N, because of the maximality of H. Then, there exists a non-zero element y in $N\cap\{H, x\}$, and y is expressible as

$$y=k+mx$$

where h is an element of H, and m is some integer. We have $(m, p)=1$, for px belongs to H and $N\cap H=0$. There exist some integers s and t

such that $ms+p^e t=1$, where e is the exponent of x. Thus we have

$$smx+tp^e x=x\ ,$$

whence it follows

$$sy-sh=x\ .$$

Owing to $\gamma=\beta+1\leqq\alpha+1$, y is contained in $N\subseteq p^\alpha G\subseteq p^\beta G$, and we have

$$sh=sy-x\in p^\beta G\ .$$

Hence, by the inductive hypothesis, it holds

$$sh\in H\cap p^\beta G=p^\beta H\ .$$

Since $psy=a+psh$ belongs to $N\cap H=0$, we have

$$a=-psh\in p(p^\beta H)\ .$$

Therefore, a is an element of $p(p^\beta H)=p^\gamma H$. Thus we have proved $p^\gamma H=p^\gamma G\cap H$ when γ is not a limit ordinal.

If γ is a limit ordinal, we have, by the inductive assumption,

$$\begin{aligned} H\cap p^\gamma G &= H\cap\Big(\bigcap_{\beta<\gamma} p^\beta G\Big) \\ &= \bigcap_{\beta<\gamma}(H\cap p^\beta G) \\ &= \bigcap_{\beta<\gamma} p^\beta H \\ &= p^\gamma H\ . \end{aligned}$$

This completes the proof of the theorem.

In order to give the another proof, we prove the following proposition.

PROPOSITION. *Let H be an N-high subgroup of an Abelian p-group G where N is a subgroup of $p^\alpha G$. Then $H\cap p^\beta G$ is an N-high subgroup of $p^\beta G$, for all $\beta\leqq\alpha$.*

PROOF. To see that $H\cap p^\beta G$ is an N-high subgroup of $p^\beta G$, suppose that there exists an element x in $p^\beta G$ with $\{H\cap p^\beta G, x\}\cap N=0$, and $x\notin H\cap p^\beta G$. Therefore, x dose not belong to H. By the assumption for H, $\{H, x\}$ intersects with N. Thus we have a non-zero element g in N with $g=h+mx(h\in H)$ with some integer m. Owing to

$$g\in N\subseteq p^\alpha G\subseteq p^\beta G,$$

$h=g-mx$ must be an element of $H\cap p^\beta G$. This contradicts to $\{H\cap p^\beta G, x\}\cap N=0$.

Making use of this proposition, we shall obtain another proof of our previous theorem.

We assume the statement true for every ordinal $\beta<\gamma$, where γ is any ordinal $0<\gamma\leq\alpha+1$. When γ is a limit ordinal, the proof is quite easy, as mentioned above.

If $\gamma=\beta+1$, by the above proposition, $H\cap p^{\beta}G$ is an N-high subgroup of $p^{\beta}G$, and then it is a neat subgroup of $p^{\beta}G$. Hence we have

$$p(H\cap p^{\beta}G)=(H\cap p^{\beta}G)\cap p(p^{\beta}G) .$$

Using the inductive hypothesis, we have

$$p(p^{\beta}H)=H\cap p^{\beta}G\cap p^{\beta+1}G ,$$

namely

$$p^{\beta+1}H=H\cap p^{\beta+1}G ,$$

which completes the proof.

References

[1] P. Hill and C. Megibben, Minimal pure subgroups in primary groups, Bull. Soc. Math. France, **92** (1964), 251-257.

[2] J. M. Irwin, High subgroups of Abelian torsion groups, Pacific J. Math., **11** (1961), 1375-1384.

[3] J. M. Irwin and E. A. Walker, On N-high subgroups of Abelian groups, Pacific J. Math., **11** (1961), 1363-1374.

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